Tutorial- 3

Q.1 [Q-1/page 403/ Guler] Consider the steepest - descent method to compute a minimizer of the function $f(x,y) = x^4 + y^2 - 8y$ starting at the point $(x^0, y^0) = (0,1)$. Determine the next iteration (x_1, y_1) using several of the step- size selection rules.

Solution: According to the Steepest - Descent method, the next iteration will be

$$x^{k+1} = x^k + t^k d^k, \qquad t^k > 0,$$

where t^k is a suitable choosen quantity. and $d^k = -\nabla f(x^k)$ is steepest descent di of f at x^k.

Here, $(x^0, y^0) = (0,1)$

$$d^0 = (d_1^0, d_2^0) = -\nabla f(x^0, y^0) = -(0,-6) = (0,6), \quad \boxed{\text{where } \nabla f(x,y) = (4x^3, 2y-8)}$$

Exact Minimization Rule: Find t^0 such that

$$f\big((x^0, y^0) + t^0 (d_1^0, d_2^0)\big) \le f\big((x^0, y^0) + t(d_1^0, d_2^0)\big), \quad \forall\ t \ge 0$$

i.e $\min\limits_{t \ge 0} f\big((x^0, y^0) + t(d_1^0, d_2^0)\big)$

Now, let $g(t) = f\big((x^0, y^0) + t(d_1^0, d_2^0)\big) = f(0, 1+6t) = (1+6t)^2 - 8(1+6t) = 1 + 36t^2 + 12t - 8 - 4$
$= 36t^2 - 36t - 7,$

$$g'(t) = 72t - 36 = 0 \implies t = \frac{1}{2} \ge 0$$

$g''(t) = 72 > 0 \Rightarrow$ min of g is attained at $t^\circ = \frac{1}{2}$

$$\Rightarrow \boxed{(x^1, y^1) = (x^\circ, y^\circ) + t^\circ (d_1^\circ, d_2^\circ) = (0,1) + \frac{1}{2}(0,6) = (0,4)}$$

<u>Limited Minimization Rule:</u> choose $s = \frac{1}{2}$. Find t° which minimizes gives

$$\min_{0 \le t \le \frac{1}{3}} f\big((x^\circ, y^\circ) + t^\circ (d_1^\circ, d_2^\circ)\big)$$

$g'(t) = 0$ at $t = \frac{1}{2} \notin [0, \frac{1}{3}]$, so choose $t^\circ = \frac{1}{3}$ (as g is decreasing f^n)

$$\boxed{(x^1, y^1) = (x^\circ, y^\circ) + t^\circ (d_1^\circ, d_2^\circ) = (0,1) + \frac{1}{3}(0,6) = (0,3)}$$

<u>Constant step length Rule:</u> Let $t = \frac{1}{3}$. Then

$$\boxed{(x^1, y^1) = (x^\circ, y^\circ) + \frac{1}{3}(d_1^\circ, d_2^\circ) = (0,3)}$$

<u>Armijo's Rule:</u> Let $s = \frac{1}{3}$, $\beta = \sigma = \frac{1}{2}$. Then, test the inequality

$$f(x^\circ, y^\circ) - f\big((x^\circ, y^\circ) + \beta^i s (d_1^\circ, d_2^\circ)\big) \ge -\sigma(\beta^i s) \langle \nabla f(x^\circ, y^\circ), (d_1^\circ, d_2^\circ) \rangle, \quad i = 0, 1, 2 \ldots$$

iteratively, starting with $i = 0$ and find the smallest i satisfying the inequality

Let $i = 0$.

$$f(0,1) - f\left((0,1) + \frac{1}{3}(0,6)\right) + \frac{1}{2} \times \frac{1}{3} \langle (0,-6), (0,6) \rangle = -7 - f(0,3) + \frac{1}{6}(-36)$$

$$= -7 + 15 - 6 \geq 0$$

$$\therefore \boxed{(x^1, y^1) = (x^0, y^0) + \frac{1}{3}(d_1^0, d_2^0) = (0,1) + \frac{1}{3}(0,6) = (0,3)}$$